# POLYTOPES WITH CENTRALLY SYMMETRIC FACES

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ABSTRACT

It is shown that if, for some $2 \leqq j \leqq d\text{-}2$, all the $j$-faces of a $d$-polytope $P$ are centrally symmetric, then all the faces of $P$ of every dimension are centrally symmetric.

**1. Introduction.** A well-known theorem of Aleksandrov [1] states that, if all the faces of a 3-polytope $P$ are centrally symmetric, than $P$ itself is centrally symmetric. Shephard [3] generalized this result to higher dimensions, showing that if, for some $2 \leqq j \leqq d-1$, all the $j$-faces of a $d$-polytope $P$ are centrally symmetric, then all the $k$-faces of $P$ $(j \leqq k \leqq d-1)$ and $P$ itself are centrally symmetric. (Throughout the paper we shall follow the terminology of Grünbaum [2].)

In each dimension $d \geqq 4$, it is possible to find examples of $d$-polytopes, all of whose facets ($(d-1)$-faces) are centrally symmetric, but having some $(d-2)$-faces which are not centrally symmetric. For example, let

$$P = \{(\xi_1, \cdots, \xi_d) \in E^d \mid |\xi_i| \leqq 2 \ (i = 1, \cdots, d), \ \sum_{i=1}^{d} |\xi_i| \leqq d\}.$$

Then $P$ is a certain intersection of a $d$-cube and a regular $d$-cross-polytope; its vertices are all permutations of

$$(\pm 2, \cdots, \pm 2, 0, \cdots 0) \qquad (d \text{ even}),$$

$$(\pm 2, \cdots, \pm 2, \pm 1, 0, \cdots, 0) \qquad (d \text{ odd}),$$

where $[d/2]$ of the coordinates are $0$. The facets of $P$ are of two kinds, typical examples being

$$F_1 = \{(\xi_1, \cdots, \xi_d) \in P \mid \xi_d = 2\},$$

$$F_2 = \{(\xi_1, \cdots, \xi_d) \in P \mid \sum_{i=1}^{d} \xi_i = d\};$$

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it is easy to see by considering their vertices that they are centrally symmetric. However, the $(d-2)$-face $F_1 \cap F_2$ is not centrally symmetric.

In the light of these examples, it is natural to raise the following question. Is it possible, for each $3 \leqq j \leqq d-2$, to find a $d$-polytope $P$, all of whose $j$-faces are centrally symmetric, which has some $(j-1)$-faces which are not centrally symmetric? The theorem of this paper provides a strong negative answer to this question.

THEOREM. *Let $P$ be a $d$-polytope $(d \geqq 4)$, such that for some $2 \leqq j \leqq d-2$, all the $j$-faces of $P$ are centrally symmetric. Then all the faces of $P$ of each dimension are centrally symmetric.*

**2. Proof of the theorem.** It is clearly enough to prove the following lemma, from which the theorem will follow by induction arguments on the dimensions.

LEMMA. *Let $P$ be a $d$-polytope $(d \geqq 4)$, all of whose $(d-2)$-faces are centrally symmetric, and let $F$ be a $(d-3)$-face of $P$. Then $F$ is centrally symmetric.*

We first observe that, by the theorem of Shephard [3] quoted in Section 1, the condition of the lemma implies that the facets of $P$ are centrally symmetric. (It also implies that $P$ itself is centrally symmetric, but we shall not use this fact.)

Let $G$ be any facet or $(d-2)$-face of $P$ which contains $F$. $G$ is centrally symmetric, so the reflexion in the centre of $G$ carries $F$ into a translate of $-F$. The reflexion in the centre of any facet or $(d-2)$-face of $P$ containing this translate carries it into a translate of $F$, and so on. So, to prove that $F$ is centrally symmetric, it is enough to show that there is some product of an odd number of successive such central reflexions, which takes $F$ into itself.

To do this, we proceed as follows. Let $L$ be any 3-dimensional affine subspace of $E^d$ orthogonal to aff $F$, the affine hull of the face $F$, and let $Q$ be the orthogonal projection of $P$ on to $L$. Theorem 2 of Shephard [3] asserts that $Q$ is a 3-polytope with centrally symmetric 2-faces. Further, each $k$-face of $Q$ $(0 \leqq k \leqq 2)$ corresponds to exactly one $(d-3+k)$-face of $P$, which contains a translate of $F$ or $-F$. So, for example, the two vertices of an edge of $Q$ correspond to opposite $(d-3)$-faces (one a translate of $F$, the other of $-F$) of a $(d-2)$-face of $P$.

Since $Q$ is a 3-polytope, all of whose 2-faces have an even number of edges, $Q$ must have at least one quadrilateral face. (Euler's equation implies that not all the 2-faces of $Q$ can have at least six sides; in fact $Q$ must have at least 6 quadrilateral faces.) Let $G$ be the facet of $P$ corresponding to such a quadrilateral face of $Q$, and let $G_1, G_2$ be the $(d-2)$-faces of $G$ corresponding to edges of this quadrilateral face which have a common vertex. Then $G_1 \cap G_2$ is a translate of $F$ or of $-F$, and the three successive reflexions in the centres of $G_1$, $G$ and $G_2$ take this translate into itself. That is, $F$ is centrally symmetric, which establishes the lemma.

The theorem now follows at once. For, suppose that for some $2 \leqq j \leqq d-2$, all the $j$-faces of a $d$-polytope $P$ are centrally symmetric. From the theorem of Shephard [3], all the faces of $P$ of dimension $k > j$ are centrally symmetric. If $F$ is any $(j-1)$-face of $P$, let $P_1$ be any $(j+2)$-face of $P$ (possibly $P$ itself) which contains $F$. By the lemma above applied to $P_1$ and $F$, we see that $F$ is centrally symmetric. Thus, every $(j-1)$-face of $P$ is centrally symmetric, and an induction argument on the dimension $j$ shows at once that all the faces of $P$ of dimension $k < j$ are also centrally symmetric. This completes the proof of the theorem.

It may be remarked that a slightly different proof of the theorem could be constructed by considering the orthogonal projection of the $d$-polytope $P$, all of whose $j$-faces $(2 \leqq j \leqq d-2)$ are centrally symmetric, on to a $(d-j+1)$-dimensional affine subspace orthogonal to the affine hull of any $(j-1)$-face of $P$. Theorem 2 of Shephard [3] again asserts that all the faces of this $(d-j+1)$-polytope are centrally symmetric, and the argument of the lemma applied to any quadrilateral face of the polytope shows at once that the original $(j-1)$-face of $P$ is centrally symmetric.

## References

1. A. D. Aleksandrov, *A theorem on convex polyhedra*, Trudy Mat. Inst. Steklov, **4** (1933), 87. (Russian.)

2. B. Grünbaum, *Convex Polytopes*, John Wiley and Sons, London–New York–Sydney, 1967.

3. G. C. Shephard, *Polytopes with centrally symmetric faces*, Canad. J. Math., **19** (1967), 1206–1213.

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